Mathematical Control Theory as Explained by the Rocket Car

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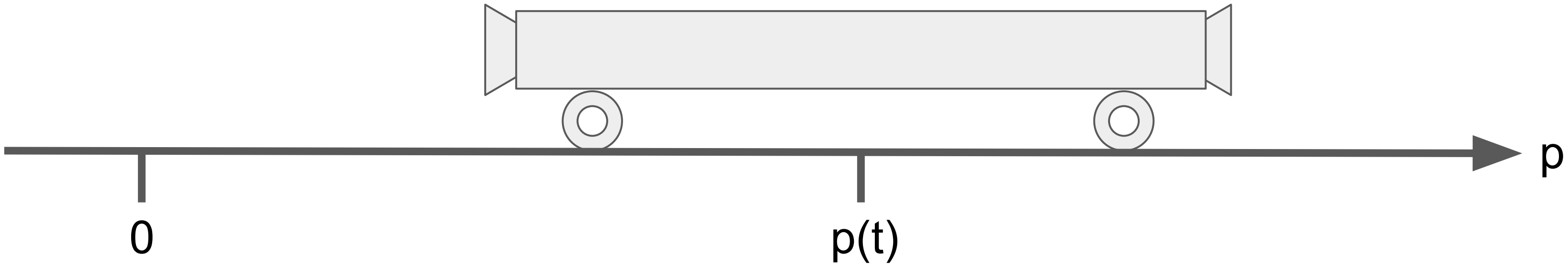
The Rocket Car

- Car running on level rails, with two rocket engines (one on each end)
  - Car has mass 1
- Problem: move the car from an initial location to a fixed destination
  - Destination is always placed at origin for simplicity



What does a control problem look like?

System: the situation that we wish to examine and control

Dynamics: how the state changes under the influence of controls

State: characteristics of the system that we wish to govern

Constraints: practical limitations on controls

Control: used to influence the state

Objective: ideal or desired “target state”

What about the Rocket Car?

System: car + track

State: $\mathbf{x}(t) = (p(t), \dot{p}(t))$

$$\mathbf{x}(0) = (p_0, v_0)$$

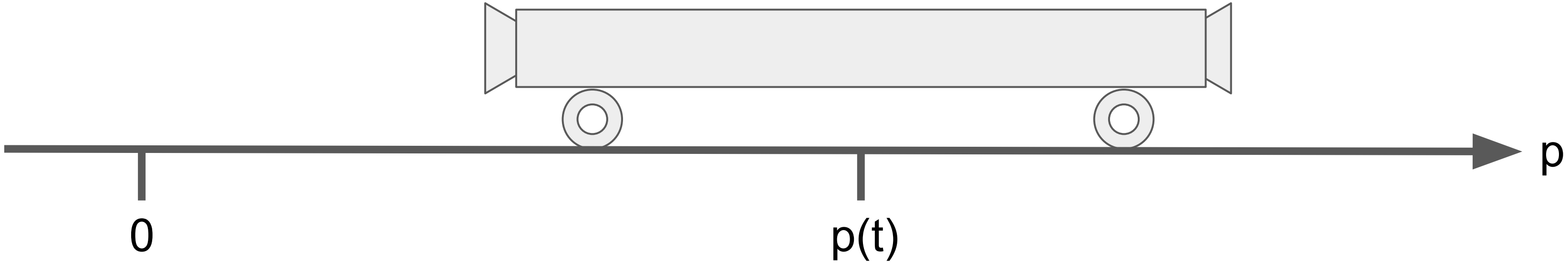
Control: real-valued function u(t) representing the force due to engines firing

Dynamics: given by Newton's Law,

$$\ddot{p}(t) = u(t) \Rightarrow \mathbf{x}(t) = \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix}, \quad \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + u(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Constraints: u(t) measurable and bounded, $|u(t)| \leq 1$ for convenience

Objective: T(t) = (0,0)



How do we define an optimal control?

- In the case that a control problem has multiple **successful controls**, we must find an **optimal control**

- **Cost/performance criteria**: used to motivate choice of one control over the other
  - Least time
  - Least energy expended
  - Least fuel expended

Setting up our problem

- Assumption: dynamics of the system are determined by a vector ODE

$$\dot{\mathbf{x}} = \mathbf{f}\left(t, \mathbf{x}(t), \mathbf{u}(t)\right), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

- The solution of our ODE for a given **u**(t) is called the **response** to **u**(t)

$$\mathbf{x}[t] \equiv \mathbf{x}(t, \mathbf{x}_0, \mathbf{u}(\cdot))$$

- **Control problem**: determine $\mathbf{x}_0$ and $u(\cdot) \in U_m$ which satisfy $\mathbf{x}[t_1] \in T(t_1)$ for some $t_1 > 0$

Controllability

Reachable set: the set of states which can be reached at time *t*

$$K(t;\mathbf{x}_0) = \{\mathbf{x}(t;\mathbf{x}_0,\mathbf{u}(\cdot\,))|\mathbf{u}(\cdot\,) \in U_m\}$$

Reachable cone: plot of reachable sets found by varying *t*

$$RC(\mathbf{x}_0) = \{(t,\mathbf{x}(t;\mathbf{x}_0,\mathbf{u}(\cdot\,)))|t \geq 0, \mathbf{u}(\cdot\,) \in U_m\}$$

Controllable set: the set of initial states for which at least one successful control exists

$$C(t_1) = \{\mathbf{x}_0 \in \mathbb{R}^n|\exists\mathbf{u}(\cdot\,) \in U_m \; s.\,t.\; \mathbf{x}(t_1;\mathbf{x}_0,\mathbf{u}(\cdot\,) \in T(t_1)\}$$

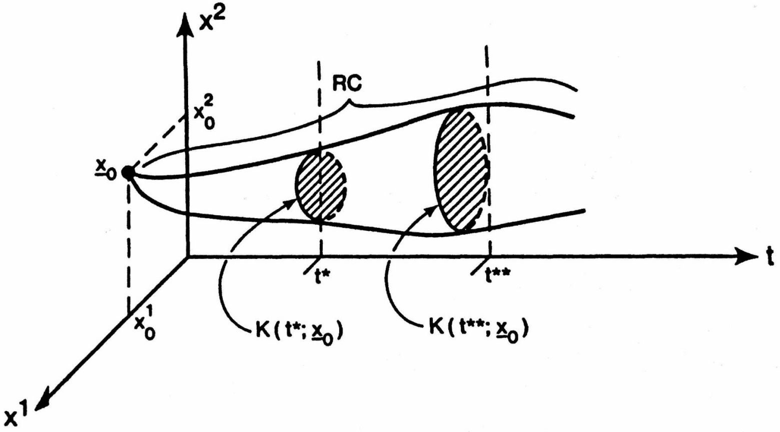


Figure 4

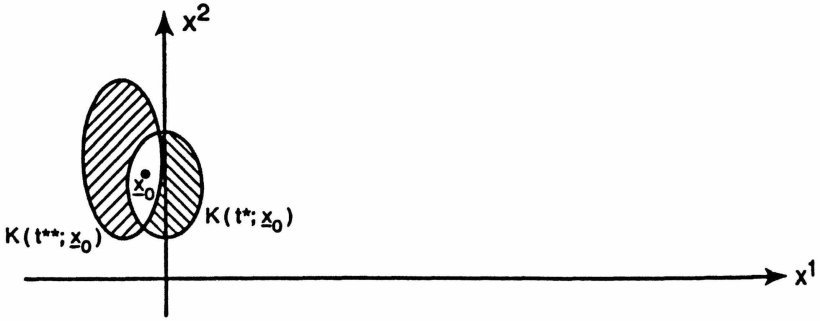


Figure 5

Linear Autonomous Case

- First we will examine the Linear Autonomous case, where A and B are constant matrices: $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$
- For a given **u**(.) ∈ U_m and initial state $\mathbf{x}_0$, the **response formula** is given as follows:

$$\mathbf{x}[t] \equiv \mathbf{x}(t; \mathbf{x}_0, \mathbf{u}(\cdot)) = \mathbf{X}(t)\mathbf{X}^{-1}(0)\mathbf{x}_0 + \int_0^t \mathbf{X}(t)\mathbf{X}^{-1}(s)\mathbf{B}(s)\mathbf{u}(s)ds$$

- Main results:
  - the controllable set C is arcwise connected, symmetric, and convex
  - C is open ⇔ the target **0** ∈ Int(C) ⇔ rank(M) = n
    M: controllability matrix defined by M = {B, AB,..., A^{n-1}B}
  - C = R^n iff rank(M) = n and no eigenvalue of A has positive real part

General Problem

- Now let's look at the general case,

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}), \;\; \mathbf{x}(0) = \mathbf{x}_0, \;\; \mathbf{u}(\cdot) \in U_m$$

$$C[\mathbf{u}(\cdot)] = \int_0^{t_1} f^0(t, \mathbf{x}[t], \mathbf{u}(t))dt$$

where **f** and f^0 are continuous functions

- Sufficient conditions for an optimal control to exist
  - If the set of successful controls is nonempty and such controls satisfy an a priori bound, and in addition the set of points

$$\hat{\mathbf{f}}(t, \mathbf{x}, \Omega) = \{\left(f^0(t, \mathbf{x}, \mathbf{v}), \mathbf{f}^T(t, \mathbf{x}, \mathbf{v})\right)^T \mathbf{v} \in \Omega\}$$

  is a convex set, then there exists an optimal control.

Rocket Car Problem

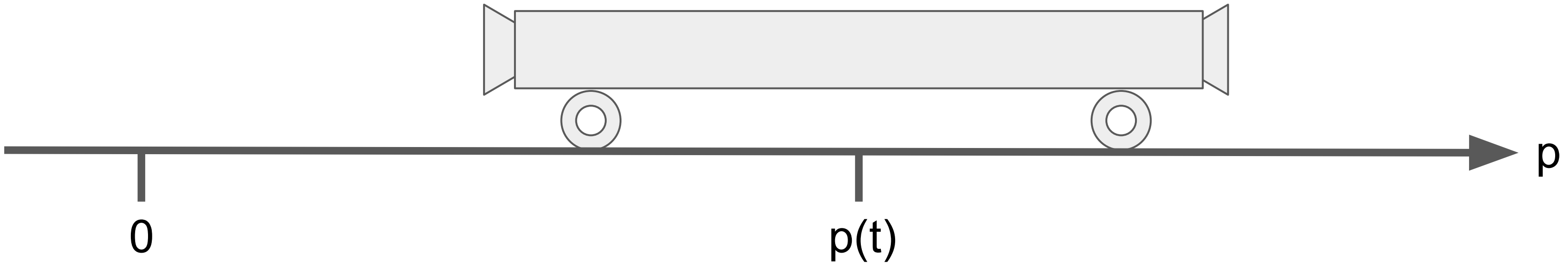
- To simplify our search for an optimal control, we consider controls in

$$U_{BB}[0,t_1] = \{\mathbf{u}(\cdot) \in U_m[0,t_1] \,|\, |u^i(t)| \equiv 1, i = 1,\ldots,m; \mathbf{u}(\cdot) \text{ piecewise constant on } [0,t_1]\}$$

- As usual, we take T(t) = **0** and define a cost function:

$$C[u(\cdot)] = \int_0^{t_1} (\lambda_1 + \lambda_2[q(t)]^2 + \lambda_3|u(t)|)dt$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$



Rocket Car Problem

- For the case when u(.) is fixed at -1 or 1, the responses fall on parabolas:

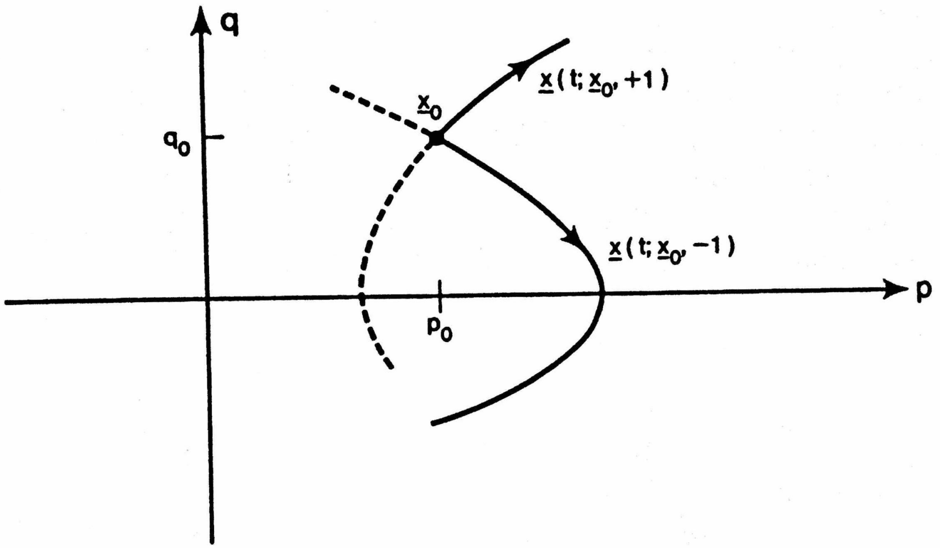


Figure 6

Rocket Car Problem

- When we let u(.) change sign once, the reachable set looks like a football:

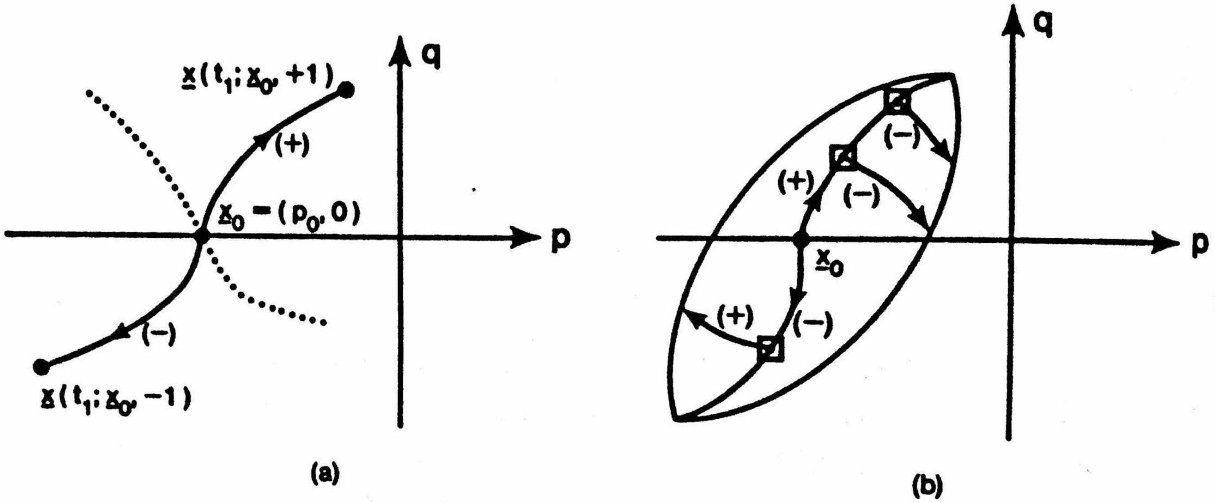


Figure 8

- In fact, this is the reachable set for t_1 and all general controls

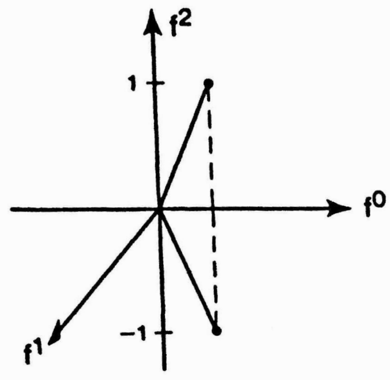
Rocket Car Problem

- To verify our theorems for the linear autonomous and general cases:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \; M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

rank(M) = n and A has eigenvalues 0,0 so C = R^n

- The graph of $\hat{\mathbf{f}}(t, \mathbf{x}, \Omega) = \{\left(f^0(t, \mathbf{x}, \mathbf{v}), \mathbf{f}^T(t, \mathbf{x}, \mathbf{v})\right)^T \mathbf{v} \in \Omega\}$ is a pair of broken line segments with slope ±1/λ_3, so theorem only holds when λ_3 = 0 (when we ignore fuel consumption)



(a)

So what is the solution?

- It depends on our cost function!

$$C[u(\cdot)] = \int_0^{t_1} (\lambda_1 + \lambda_2 [q(t)]^2 + \lambda_3 |u(t)|) dt$$

- For the time optimal problem, i.e. $\lambda_1=1$, $\lambda_2=\lambda_3=0$ the optimal control is bang-bang
- For the minimum fuel problem, i.e. $\lambda_1=\lambda_2=0$, $\lambda_3=1$ there is no optimal control

Thank you!

- Patrick
- Dr. Merling and Dr. Brown
- JHU Math Department